THE SEVERI INEQUALITY $K^2 \geq 4\chi$ FOR SURFACES OF MAXIMAL ALBANESE DIMENSION

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ABSTRACT. We prove the so-called Severi inequality, stating that the invariants of a minimal smooth complex projective surface of maximal Albanese dimension satisfy:

$$K_S^2 \geq 4\chi(S).$$

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1. INTRODUCTION

In this paper we prove the so-called Severi inequality:

If S is a minimal smooth complex projective surface of maximal Albanese dimension, then $K_S^2 \geq 4\chi(S)$.

This inequality has a long history. Severi ([Se]) stated it as a theorem in 1932, but his proof was not correct (cf. [Ca]). Around the end of the 1970's the inequality was posed as a conjecture by Reid ([Re]) and by Catanese ([Ca]). Motivated by this conjecture, Xiao wrote the foundational paper [Xi] on surfaces fibred over a curve, in which he proves the conjecture in the special case of a surface admitting a fibration over a curve of positive genus. Finally, at the end of the 1990's, the conjecture was almost solved by Manetti ([Ma]), who proved it under the additional assumption that the surface have ample canonical bundle. His proof is based on a very fine study of the positivity properties of the tautological line bundle of $\mathbb{P}(\Omega^1_S \otimes \omega_S)$ and it allows him also to classify the surfaces for which equality holds, still under the assumption that the canonical bundle be ample.

Our proof of the conjecture in the general case is completely different from Manetti's. We derive the Severi inequality from a well known inequality for the slope of a fibered surface proven by Xiao in the same paper [Xi] and, independently, by Cornalba–Harris ([CH]) in the semistable case. In order to do this, we construct first an infinite

sequence of fibred surfaces $f_d\colon Y_d \to \mathbb{P}^1$ such that the slope of f_d converges to the ratio $K_S^2/\chi(S)$ as d goes to infinity (cf. Prop. 2.3). Then we obtain the Severi inequality by applying the slope inequality to the fibrations f_d and taking the limit for $d \to \infty$.

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Notation and conventions. We work over the field of complex numbers; all varieties are projective. Linear equivalence is denoted by $\equiv$ and numerical equivalence by $\sim_{num}$. We say that a surface S has maximal Albanese dimension if the image of its Albanese map is a surface.

The remaining notation is standard in algebraic geometry. We just recall the numerical invariants associated to a surface S: the self intersection K_S^2 of the canonical divisor K_S, the *geometric genus* $p_g(S) := h^0(S, K_S)$, the *irregularity* $q(S) := h^0(S, \Omega^1_S)$ and the *Euler–Poincaré characteristic* $\chi(S) := p_g(S) - q(S) + 1$.

2. The Severi inequality

In this section we prove the following:

Theorem 2.1 (Severi inequality)**.** *Let S be a smooth minimal complex surface of maximal Albanese dimension. Then:*

$$K_S^2 \geq 4\chi(S).$$

We begin by recalling some facts about surfaces fibred over a curve.

Let Y be a smooth complex surface and let $f\colon Y \to B$ be a fibration onto a smooth curve of genus b. Assume that f is relatively minimal, i.e., that no fibre of f contains a $-1-$curve, and that a general fibre of f has genus $g \geq 2$. It is well known (cf. [Be]) that one has:

$$\chi(Y) \geq (b-1)(g-1)$$

with equality holding if and only if the fibration f is smooth and isotrivial. (Recall that a fibration is said to be isotrivial if all its smooth fibres are isomorphic).

If f is not smooth and isotrivial, then one defines (cf. [Xi]) the *slope* $\lambda(f)$ of f as:

$$\lambda(f) := \frac{K_Y^2 - 8(b-1)(g-1)}{\chi(Y) - (b-1)(g-1)}.$$

The fundamental inequality for the slope of a fibration is the following:

Theorem 2.2 (Xiao, Cornalba–Harris)**.** *Let $f\colon Y \to B$ be a relatively minimal fibration, not smooth and isotrivial, with fibres of genus $g \geq 2$. Then one has the following inequalities for the slope $\lambda(f)$ of f:*

$$4(g-1)/g \leq \lambda(f) \leq 12.$$

Proof. See [Xi], Theorem 2. For the case of a semistable fibration, see also [CH]. □

We are going to reduce the proof of the Severi inequality in the general case to Theorem 2.2 by means of the following:

Proposition 2.3. *Let S be a smooth minimal complex surface of general type and of maximal Albanese dimension. There exists a sequence of smooth complex surfaces Y_d, $d \in \mathbb{N}_{>0}$, and relatively minimal fibrations $f_d\colon Y_d \to \mathbb{P}^1$ such that:*

(i) *f_d is not isotrivial;*
(ii) *$\lim_{d\to\infty} g_d = +\infty$, where g_d is the genus of a general fibre of f_d;*
(iii) *$\lim_{d\to\infty} \lambda(f_d) = K_S^2/\chi(S)$.*

Proof. Let $a\colon S \to A$ be the Albanese map and let $q \geq 2$ be the irregularity of S. Let L be a very ample line bundle on A, let H be the pull-back of L on S and set $\alpha := H^2$, $\beta := K_S H$. Fix an integer $d \geq 1$, let $\mu\colon A \to A$ be the multiplication by d and consider the following cartesian diagram:

$$\begin{array}{ccc} S' & \xrightarrow{p} & S \\ {\scriptstyle a'}\downarrow & & \downarrow{\scriptstyle a} \\ A & \xrightarrow{\mu} & A \end{array} \tag{2.1}$$

The surface S' is smooth minimal of maximal Albanese dimension with invariants:

$$K_{S'}^2 = d^{2q} K_S^2, \quad \chi(S') = d^{2q}\chi(S). \tag{2.2}$$

By [LB, Ch. 2, Prop. 3.5], one has the following equivalence on A:

$$\mu^* L \sim_{num} d^2 L.$$

Hence, denoting by H' the pull-back of L on S' via the map a', one has:

$$p^* H \sim_{num} d^2 H'.$$

Using this remark, one computes:

$$H'^2 = d^{2q-4}\alpha, \quad H'K_{S'} = d^{2q-2}\beta. \tag{2.3}$$

Now let $D_1, D_2 \in |H'|$ be general smooth curves. Set $C_1 := D_1 + D_2$ and let $C_2 \in |2H'|$ be a general curve, so that C_2 is smooth and C_1 and C_2 intersect transversely at $4d^{2q-4}\alpha$ points. Let $\varepsilon\colon Y_d \to S'$ be the blow up of the intersection points of C_1 and C_2. The invariants of Y_d are the following:

$$K^2_{Y_d} = K^2_{S'} - 4H'^2 = d^{2q}K^2_S - 4d^{2q-4}\alpha \tag{2.4}$$

$$\chi(Y_d) = \chi(S') = d^{2q}\chi(S) \tag{2.5}$$

The pencil spanned by C_1 and C_2 defines a fibration $f_d\colon Y_d \to \mathbb{P}^1$ which is relatively minimal by construction. Furthermore, f_d is not isotrivial, since the strict transform of C_1 is a singular stable fibre. The genus g_d of a general fibre of f_d is equal to the genus of C_2, namely one has:

$$g_d = 1 + K_{S'}H' + 2H'^2 = 1 + d^{2q-2}\beta + 2d^{2q-4}\alpha. \tag{2.6}$$

We have $\beta = K_S H > 0$, since S is minimal of general type and $|H|$ is free, and $q > 1$, since S has maximal Albanese dimension. By these remarks, statement (ii) follows from (2.6).

The slope of f_d is given by the following expression:

$$\lambda(f_d) = \frac{K^2_{Y_d} + 8(g_d - 1)}{\chi(Y_d) + (g_d - 1)} \tag{2.7}$$

Hence statement (iii) follows by (2.4), (2.5) and (2.6). □

Proof of Thm. 2.1. If S is not of general type, then the inequality follows from the classification of surfaces, hence we may assume that S is of general type.

Consider the surfaces Y_d of Proposition 2.3. By Theorem 2.2, we have:

$$\lambda(f_d) \geq 4(g_d - 1)/g_d \quad \forall d \geq 1. \tag{2.8}$$

By Proposition 2.3, taking the limit of (2.8) for $d \to \infty$ we get:

$$K^2_S \geq 4\chi(S).$$

□

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